

THE FIRST STEKLOFF EIGENVALUE IN WEIGHTED RIEMANNIAN MANIFOLDS

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ABSTRACT. In study of eigenvalue problems, a classical problem is the Stekloff eigenvalue problem. There are many estimates of the first non-zero Stekloff eigenvalue, including a sharp estimate on surfaces, obtained by Escobar in [4]. In this paper we are interested to study this problem in weighted context. Various estimates are obtained, including a sharp estimate on surfaces, similar to that Escobar obtained.

1. INTRODUCTION

The Classical Stekloff problem in its original form is the eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

was introduced by him in [21] for bounded domains Ω of the plane and afterward this was studied by Payne in [16] for bounded domains in the plane with non-negative curvature. This problem has a physical interesting because the eigenfunctions represents the steady state temperature on a domain and the flux on the boundary is proportional to the temperature, see [21] for more details. Thenceforth many authors have studied about this subject and they had many advances on the understanding of this topic, see for instance [21, 22, 16, 4, 5, 6, 12, 14, 18, 23, 24, 3, 8] and references therein. More specifically, many authors studied ways to estimate or determine exactly the eigenvalues associated with the Stekloff problem and modifications of the latter, see [4, 23, 24].

Another interesting object in Differential Geometry are the Riemannian manifolds endowed with a smooth positive density function, this is directly related to Ricci flow, mean curvature flow, theory of optimal transportation, see [15, 7] for a good overview of this subject. Aiming study the various Stekloff's problems in the weighted context, we will introduce the necessary concepts. We recall that a weighted Riemannian manifold is a Riemannian manifold (M, g) endowed with a real-valued smooth function $f : M \rightarrow \mathbb{R}$ which is used as a density to measure geometric objects on M . Associated

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to this structure we have an important second order differential operator defined by

$$\Delta_f u = \Delta u - \langle \nabla u, \nabla f \rangle,$$

where $u \in C^\infty$. This operator is known as Drift Laplacian.

Also, following Lichnerowich [13] and Bakry and Émery [1], the natural generalizations of Ricci curvatures are defined as

$$(1.1) \quad \text{Ric}_f = \text{Ric} + \text{Hess } f$$

and

$$(1.2) \quad \text{Ric}_f^k = \text{Ric}_f - \frac{df \otimes df}{k - n - 1},$$

where $k > n + 1$ or $k = n + 1$ and f a constant function.

In this paper we will consider M^{n+1} a compact oriented Riemannian manifold with boundary ∂M . Let $i : \partial M \hookrightarrow M$ be the standard inclusion and ν the outward unit normal on ∂M . We will denote by II its second fundamental form associate to ν , $\langle \nabla_X \nu, Y \rangle = II(X, Y)$, and by H the mean curvature of ∂M , that is, the trace of II over n .

We recall that the weighted mean curvature, introduced by Gromov in [10], of the inclusion i is given by

$$H_f = H - \frac{1}{n} \langle \nu, \nabla f \rangle.$$

Finally, consider the following three kinds of the weighted Stekloff problems:

$$(1.3) \quad \begin{cases} \Delta_f u = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu} = pu & \text{on } \partial M; \end{cases}$$

$$(1.4) \quad \begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \Delta_f u - q \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M; \end{cases}$$

$$(1.5) \quad \begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

where ν denotes the outward unit normal on ∂M . The first non-zero eigenvalues of the above problems will be denoted by p_1 and q_1 , respectively. We will use the same letter for the first non-zero eigenvalues of last two problems because whenever the weighted mean curvature of ∂M is constant then the problems are equivalents.

Lastly, for the sake of simplicity, we will omit the weighted volume element in the integrals in all text. Now, we are able to introduce our results.

Our first result reads as follows:

Theorem 1.1. *Let M^{n+1} be a compact weighted Riemannian manifold with $\text{Ric}_f^k \geq 0$ and boundary ∂M . Assume that the weighted mean curvature of ∂M satisfies $H_f \geq \frac{(k-1)c}{n}$, to some positive constant c , and that second fundamental form $II \geq cI$, in the quadratic form sense. Denote by λ_1 the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M . Let p_1 be the first eigenvalue of the weighted Stekloff eigenvalue problem (1.3). Then,*

$$(1.6) \quad p_1 \leq \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2})$$

with equality occurs if and only if M is isometric to an n -dimensional euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n + 1$.

The second result is the following:

Theorem 1.2. *Let M^{n+1} be a compact connected weighted Riemannian manifold with $\text{Ric}_f^k \geq 0$ and boundary ∂M . Assume that the weighted mean curvature of ∂M satisfies $H_f \geq \frac{k-1}{k}c$, to some positive constant c . Let q_1 be the first eigenvalue of the weighted Stekloff eigenvalue problem (1.4). Then*

$$q_1 \geq nc.$$

Moreover, equality occurs if and only if M is isometric to a euclidean ball of radius $\frac{1}{c}$ in \mathbb{R}^{n+1} , f is constant and $k = n + 1$.

The next results are

Theorem 1.3. *Let M^{n+1} be a compact connected weighted Riemannian manifold with boundary ∂M . Denote by A , V the weighted area of ∂M and the weighted volume of M , respectively. Let q_1 be the first eigenvalue of the weighted Stekloff eigenvalue problem (1.4). Then,*

$$q_1 \leq \frac{A}{V}.$$

Moreover, if in addition that the Ric_f^k of M is non-negative and that there is a point $x_0 \in \partial M$ such that $H_f(x_0) \geq \frac{A}{(n+1)V}$, and $q_1 = \frac{A}{V}$ implies that M is isometric to an $(n+1)$ -dimensional Euclidean ball, f is constant and $k = n + 1$.

and

Theorem 1.4. *Let M^{n+1} be a compact connected weighted Riemannian manifold with $\text{Ric}_f^k \geq 0$ and boundary ∂M nonempty. Assume that $H_f \geq \frac{(k-1)c}{n}$, for some positive constant c . Let q_1 be the first eigenvalue of the problem (1.5). Then*

$$q_1 \geq c.$$

Moreover, equality occurs if and only if M is isometric to a ball of radius $\frac{1}{c}$ in \mathbb{R}^{n+1} , f is constant and $k = n + 1$.

Lastly, we announce a sharp estimate of the first non-zero Stekloff eigenvalue of surfaces about suitable hypotheses.

Theorem 1.5. *Let M^2 be a compact weighted Riemannian manifold with boundary. Assume that M has non-negative Ric_f , and that the geodesic curvature of ∂M , k_g satisfies $k_g - f_\nu \geq c > 0$. Let p_1 be the first non-zero eigenvalue of the Stekloff problem (1.3). Assume that f is constant on the boundary ∂M , then $p_1 \geq c$. Moreover, the equality occur if and only if M is the Euclidean ball of radius c^{-1} and f is constant.*

2. PRELIMINARIES

In this section we recall some necessary results to prove the theorems enunciated in the introduction. We will present some proofs for the sake of completeness.

In [2] the authors proved the following useful inequality.

Proposition 2.1. *Let u be a smooth function on M^{n+1} . then we have*

$$|\text{Hess } u|^2 + \text{Ric}_f(\nabla u, \nabla u) \geq \frac{(\Delta_f u)^2}{k} + \text{Ric}_f^k(\nabla u, \nabla u),$$

for every $k > n + 1$ or $k = n + 1$ and f is a constant. Moreover, equality holds if and only if $\text{Hess } u = \frac{\Delta u}{n+1} \langle \cdot, \cdot \rangle$ and $\langle \nabla u, \nabla f \rangle = -\frac{k-n-1}{k} \Delta_f u$ ¹.

Proof. Let $\{e_1, \dots, e_{n+1}\}$ be a orthonormal basis of $T_p M$, then by Cauchy-Schwarz inequality we have that

$$(2.1) \quad (\Delta u)^2 \leq (n+1)|\text{Hess } u|^2.$$

Using that $\frac{1}{n+1}a^2 + \frac{1}{k-n-1}b^2 \geq \frac{1}{k}(a-b)^2$ with equality if and only if

$$(2.2) \quad a = -\frac{(n+1)b}{k-n-1},$$

we obtain

$$(2.3) \quad \begin{aligned} |\text{Hess } u|^2 + \text{Ric}_f(\nabla u, \nabla u) &\geq \frac{1}{n+1}(\Delta u)^2 + \text{Ric}_f^k(\nabla u, \nabla u) + \frac{\langle \nabla f, \nabla u \rangle^2}{k-n-1} \\ &\geq \frac{1}{k}(\Delta u - \langle \nabla f, \nabla u \rangle)^2 + \text{Ric}_f^k(\nabla u, \nabla u) \\ &= \frac{1}{k}(\Delta_f u)^2 + \text{Ric}_f^k(\nabla u, \nabla u). \end{aligned}$$

If the equality holds, then since we use the Cauchy-Schwarz's inequality in (2.1) we obtain that $\text{Hess } u = \lambda \langle \cdot, \cdot \rangle$, and by (2.2)

$$\Delta u = -\frac{(n+1)\langle \nabla f, \nabla u \rangle}{k-n-1},$$

¹This term only appear in the case of a non constant function.

Consequently

$$\Delta_f u = -\frac{(n+1)\langle \nabla f, \nabla u \rangle}{k-n-1} - \langle \nabla f, \nabla u \rangle = -\frac{k}{k-n-1} \langle \nabla f, \nabla u \rangle.$$

The converse is immediate. \square

In [14] the authors showed that, for a smooth function u defined on an n -dimensional compact weighted manifold M with boundary ∂M , the following identity holds if $h = \frac{\partial u}{\partial \nu}$, $z = u|_{\partial M}$ and Ric_f denotes the generalized Ricci curvature of M :

$$\begin{aligned} (2.4) \quad \int_M [(\Delta_f u)^2 - |\text{Hess } u|^2 - \text{Ric}_f(\nabla u, \nabla u)] &= \\ &= \int_{\partial M} [n H_f h^2 + 2h \bar{\Delta}_f z + II(\bar{\nabla} z, \bar{\nabla} z)]. \end{aligned}$$

Here, $\bar{\Delta}$ and $\bar{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M , respectively.

Using the Proposition 2.1 we have that

$$\begin{aligned} (2.5) \quad \int_M \frac{k-1}{k} [(\Delta_f u)^2 - \text{Ric}_f^k(\nabla u, \nabla u)] &\geq \\ &\geq \int_{\partial M} [n H_f h^2 + 2h \bar{\Delta}_f z + II(\bar{\nabla} z, \bar{\nabla} z)]. \end{aligned}$$

The next result is an estimate for the first non-zero eigenvalue of the Drift Laplacian on closed submanifolds. This result is a slight modification of Theorem 1.6 in [11] and it reads as follows.

Proposition 2.2. *Let M^{n+1} be a compact weighted Riemannian manifold with nonempty boundary ∂M and $\text{Ric}_f^k \geq 0$. If the second fundamental form of ∂M satisfies $II \geq cI$, in the quadratic form sense, and $H_f \geq \frac{k-1}{n}c$, then*

$$\lambda_1(\partial M) \geq (k-1)c^2,$$

where λ_1 is the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M . The equality holds if and only if M is isometric to an Euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n+1$.

Proof. Let z be an eigenfunction corresponding to the first non-zero eigenvalue λ_1 of the Drift Laplacian of ∂M , that is,

$$(2.6) \quad \bar{\Delta}_f z + \lambda_1 z = 0.$$

Let $u \in C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u = z & \text{on } \partial M. \end{cases}$$

It then follows from (2.5) and the non-negativity of Ric_f^k of M that

$$(2.7) \quad 0 \geq \int_{\partial M} [nH_f h^2 + 2h\bar{\Delta}_f z + II(\bar{\nabla}z, \bar{\nabla}z)].$$

Since $II \geq cI$, we have

$$II(\bar{\nabla}z, \bar{\nabla}z) \geq c|\bar{\nabla}z|^2,$$

and noticing that

$$\int_{\partial M} |\bar{\nabla}z|^2 = - \int_{\partial M} z\bar{\Delta}z = \lambda_1 \int_{\partial M} z^2,$$

we obtain

$$\begin{aligned} 0 &\geq \int_{\partial M} [nH_f h^2 + 2h\bar{\Delta}_f z + II(\bar{\nabla}z, \bar{\nabla}z)] \\ &\geq \int_{\partial M} [(k-1)ch^2 - 2\lambda_1 zh + c\lambda_1 z^2] \\ &= \int_{\partial M} \left[(k-1)c \left(h - \frac{\lambda_1 z}{(k-1)c} \right)^2 + \lambda_1 \left(c - \frac{\lambda_1}{(k-1)c} \right) z^2 \right] \\ &\geq \lambda_1 \left(c - \frac{\lambda_1}{(k-1)c} \right) \int_{\partial M} z^2. \end{aligned}$$

Consequently,

$$\lambda_1 \geq (k-1)c^2,$$

which proof the first part of theorem. The equality case follows by Proposition 2.1 and a careful analysis in the equalities that occur. The converse is immediate. \square

Recall the following version of Hopf boundary point lemma, see its proof in [9], Lemma 3.4.

Proposition 2.3 (Hopf boundary point lemma). *Let (M^n, g) be a complete Riemannian manifold and let $\Omega \subset M$ be a closed domain. If $u : \Omega \rightarrow \mathbb{R}$ is a function with $u \in C^2(\text{int}(\Omega))$ satisfying*

$$\Delta u + \langle X, \nabla u \rangle \geq 0,$$

where X is a bounded vector field, $x_0 \in \partial\Omega$ is a point where

$$u(x) < u(x_0) \quad \forall x \in \Omega,$$

u is continuous at x_0 , and Ω satisfies the interior sphere condition at x_0 , then

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

if this outward normal derivative exists.

3. PROOF OF THE SHARP BOUNDS FOR THE STEKLOFF EIGENVALUES

In this section we will give the proof of the four first results announced in introduction and for this we will use all tools presented in the preliminaries.

Proof of Theorem 1.1. Let u be the solution of the following problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u|_{\partial M} = z, \end{cases}$$

where z is a first eigenfunction on ∂M corresponding to λ_1 , that is, z satisfies $\Delta_f z + \lambda_1 z = 0$ on ∂M . Set $h = \frac{\partial u}{\partial \nu}|_{\partial M}$, then we have from the Rayleigh inequality that (cf.[12])

$$(3.1) \quad p_1 \leq \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2}$$

and

$$(3.2) \quad p_1 \leq \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2}$$

Notice that (3.2) it is the variational principle, and (3.1) it is obtained as follows,

$$\begin{aligned} p_1 &\leq \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} = \frac{-\int_M u \Delta_f u + \int_{\partial M} u \langle \nabla u, \nu \rangle}{\int_{\partial M} z^2} \\ &= \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} \cdot \frac{\int_{\partial M} u \langle \nabla u, \nu \rangle}{\int_M |\nabla u|^2} \\ &= \frac{1}{\int_{\partial M} z^2} \cdot \frac{(\int_{\partial M} u \langle \nabla u, \nu \rangle)^2}{\int_M |\nabla u|^2} \\ &\leq \frac{\int_{\partial M} z^2}{\int_{\partial M} z^2} \cdot \frac{\int_{\partial M} \langle \nabla u, \nu \rangle^2}{\int_M |\nabla u|^2} \\ &= \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2}, \end{aligned}$$

which gives

$$(3.3) \quad p_1^2 \leq \frac{\int_{\partial M} h^2}{\int_{\partial M} z^2}.$$

It then follows by substituting u into the equation (2.4), and using the Proposition 2.1, that

$$\begin{aligned}
 (3.4) \quad 0 &\geq \int_M \frac{k-1}{k} [(\Delta_f u)^2 - \text{Ric}_f^k(\nabla u, \nabla u)] \geq \\
 &\geq \int_{\partial M} [nH_f h^2 + 2h\bar{\Delta}_f z + II(\bar{\nabla} z, \bar{\nabla} z)] \\
 &\geq \int_{\partial M} [(k-1)ch^2 - 2\lambda_1 z + c|\bar{\nabla} z|^2].
 \end{aligned}$$

Note that, by Green's formula,

$$\int_{\partial M} |\bar{\nabla} z|^2 = \int_{\partial M} \langle \bar{\nabla} z, \bar{\nabla} z \rangle = - \int_{\partial M} z \bar{\Delta}_f z = \lambda_1 \int_{\partial M} z^2.$$

Putting this expression in (3.4) we have that

$$\begin{aligned}
 0 &\geq (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \int_{\partial M} hz + c\lambda_1 \int_{\partial M} z^2 \\
 &\geq (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} + c\lambda_1 \int_{\partial M} z^2 \\
 &= \frac{(k-1)c^2 - \lambda_1}{c} \int_{\partial M} h^2 + \left[\sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} \right]^2,
 \end{aligned}$$

from where

$$\frac{\sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \geq \sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}$$

and

$$\frac{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \leq \sqrt{c\lambda_1} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}},$$

that is,

$$\begin{aligned}
 \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} &\leq \frac{c\sqrt{\lambda_1}}{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} \\
 &= \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Using (3.3), we obtain

$$p_1 \leq \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2} \right).$$

Now, assume that

$$p_1 = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2} \right).$$

So, we also have that

$$\left(\int_{\partial M} h^2\right)^{\frac{1}{2}} = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right) \left(\int_{\partial M} z^2\right)^{\frac{1}{2}}$$

and all inequalities above become equality. Thus $h = \alpha z$ and

$$\alpha = \frac{(\alpha^2 \int_{\partial M} z^2)^{\frac{1}{2}}}{(\int_{\partial M} z^2)^{\frac{1}{2}}} = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right),$$

that is,

$$h = \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) z.$$

Furthermore we infer, by Proposition 2.1, that $\text{Hess } u = 0$. Now, on the boundary ∂M , we can write

$$\begin{aligned} \nabla u &= (\nabla u)^\top + (\nabla u)^\perp \\ &= (\nabla u)^\top + \langle \nabla u, \nu \rangle \nu, \end{aligned}$$

where $(\nabla u)^\top$ is tangent to ∂M and $(\nabla u)^\perp$ is normal to ∂M . Then, take a local orthonormal fields $\{e_i\}_{i=1}^n$ tangent to ∂M . We obtain

$$\begin{aligned} 0 &= \sum_{i=1}^n \text{Hess } u(e_i, e_i) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla u, e_i \rangle \\ &= \sum_{i=1}^n \langle \nabla_{e_i} [(\nabla u)^\top + \langle \nabla u, \nu \rangle \nu], e_i \rangle \\ &= \sum_{i=1}^n \langle \nabla_{e_i} (\nabla u)^\top + \langle \nabla u, \nu \rangle \nabla_{e_i} \nu + e_i(\langle \nabla u, \nu \rangle) \nu, e_i \rangle \\ &= \bar{\Delta} z + \sum_{i=1}^n \langle \nabla u, \nu \rangle II(e_i, e_i) \\ &= \bar{\Delta} z + n H h \\ &= \bar{\Delta}_f z - f_\nu h + n H h \\ &= \bar{\Delta}_f z + n H_f h \\ &= -\lambda_1 z + c(k-1)h \\ &= -\lambda_1 z + c(k-1) \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) z, \end{aligned}$$

from where

$$\lambda_1 = (k-1)c^2.$$

Therefore, follows by Proposition 2.2 that M is isometric to an $(n+1)$ -dimensional Euclidean ball of radius $\frac{1}{c}$, f is constant and so $k = n+1$. The converse follows the ideas of the Riemannian case. \square

Proof of Theorem 1.2. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of problem (1.4), that is,

$$(3.5) \quad \begin{cases} \Delta_f^2 w = 0 & \text{in } M, \\ w = \Delta_f w - q_1 \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Set $\eta = \frac{\partial w}{\partial \nu}|_{\partial M}$; then by divergence theorem we obtain

$$\begin{aligned} \int_M (\Delta_f w)^2 &= - \int_M \langle \nabla(\Delta_f w), \nabla w \rangle + \int_{\partial M} \Delta_f w \langle \nabla w, \nu \rangle \\ &= \int_M w \Delta_f(\Delta_f w) - \int_{\partial M} w \langle \nabla(\Delta_f w), \nu \rangle + \int_{\partial M} \Delta_f w \langle \nabla w, \nu \rangle \\ &= q_1 \int_{\partial M} \eta^2, \end{aligned}$$

that is,

$$q_1 = \frac{\int_M (\Delta_f w)^2}{\int_{\partial M} \eta^2}.$$

Substituting w in (2.5), and noting that $w|_{\partial M} = z$, we have

$$\begin{aligned} \frac{k-1}{k} \int_M (\Delta_f w)^2 &\geq \int_M \text{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} n H_f \eta^2 \\ &\geq \frac{(k-1)nc}{k} \int_{\partial M} \eta^2, \end{aligned}$$

from where $q_1 \geq nc$, as we desired.

Assume now that $q_1 = nc$, then the inequalities above become equalities and consequently $H_f = \frac{k-1}{k}c$. Furthermore, we have equality in the Proposition 2.1, thus $\text{Hess } w = \frac{\Delta w}{n+1} \langle \cdot, \cdot \rangle$ and $\Delta_f w = \frac{k}{n+1} \Delta w$.

Take an orthonormal frame $\{e_1, \dots, e_n, e_{n+1}\}$ on M such that when restricted to ∂M , $e_{n+1} = \nu$. Since $w|_{\partial M} = 0$ we have

$$\begin{aligned} e_i(\eta) &= e_i \langle \nabla w, \nu \rangle \\ &= \langle \nabla_{e_i} \nabla w, \nu \rangle + \langle \nabla w, \nabla_{e_i} \nu \rangle \\ &= \text{Hess } w(e_i, \nu) + II((\nabla w)^\top, e_i) = 0, \end{aligned}$$

that is, $\eta = \rho = \text{constant}$, and so $(\Delta_f w)|_{\partial M} = q_1 \eta = nc\rho$ is also a constant. Using the fact that $\Delta_f w$ is a f -harmonic function on M , we conclude by maximum principle that $\Delta_f w$ is constant on M . Since $\Delta_f w = \frac{k}{n+1} \Delta w$, then w satisfies

$$\begin{cases} \text{Hess } w = \frac{\Delta_f w}{k} \langle \cdot, \cdot \rangle & \text{in } M, \\ w|_{\partial M} = 0. \end{cases}$$

Thus, by Lema 3 in [20], we conclude that M is isometric to a ball in \mathbb{R}^{n+1} of radius c^{-1} . Now, using the hessian of w is possible see that $w = \frac{\lambda}{2}r^2 + C$, where $\lambda = \frac{\Delta_f w}{k}$ and r is the distance function from its minimal point, see [20] for more details for this technique.

Lastly, we will show that f is constant. In fact, if $k > n+1$, then $\langle \nabla f, \nabla w \rangle$ is constant and so $f = -(k-n-1) \ln r + C$. It is a contradiction, since f is a smooth function.

□

Proof of Theorem 1.3. Now, let w be the solution of the following Drift Laplace equation

$$(3.6) \quad \begin{cases} \Delta_f w = 1 & \text{in } M, \\ w|_{\partial M} = 0. \end{cases}$$

Follows from Rayleigh characterization of q_1 that

$$(3.7) \quad q_1 \leq \frac{\int_M |\nabla w|^2}{\int_{\partial M} \eta^2} = \frac{\int_M (\Delta_f w)^2}{\int_{\partial M} \eta^2} = \frac{V}{\int_{\partial M} \eta^2},$$

where $\eta = \frac{\partial w}{\partial \nu}|_{\partial M}$. Integrating $\Delta_f w = 1$ on M and using the divergence theorem, it gives

$$V = \int_{\partial M} \eta.$$

Hence we infer from Schwarz inequality that

$$(3.8) \quad V^2 \leq A \int_{\partial M} \eta^2.$$

Consequently,

$$q_1 \leq \frac{V}{\int_{\partial M} \eta^2} \leq \frac{V}{V^2/A} = \frac{A}{V}.$$

Assume now that $\text{Ric}_f^k \geq 0$, $H_f(x_0) \geq \frac{(k-1)A}{knV}$ for some $x_0 \in \partial M$ and $q_1 = \frac{A}{V}$. In this case (3.8) become a equality and so $\eta = \frac{V}{A}$ is a constant. Consider the function ϕ on M given by

$$\phi = \frac{1}{2} |\nabla w|^2 - \frac{w}{k}.$$

Using the Bochner formula (4.1), $\Delta_f w = 1$, the Proposition 2.1 and that $\text{Ric}_f^k \geq 0$, we have that

$$(3.9) \quad \begin{aligned} \frac{1}{2} \Delta_f \phi &= |\text{Hess } w|^2 + \langle \nabla w, \nabla(\Delta_f w) \rangle + \text{Ric}_f(\nabla w, \nabla w) - \frac{1}{k} \\ &\geq \frac{1}{k} (\Delta_f w)^2 - \frac{1}{k} = 0. \end{aligned}$$

Thus ϕ is f -subharmonic. Observe that $\phi = \frac{1}{2} \left(\frac{V}{A} \right)^2$ on the boundary. In fact, if we write $\nabla w = (\nabla w)^\top + (\nabla w)^\perp$, where $(\nabla w)^\top$ is tangent to ∂M and $(\nabla w)^\perp$ is normal to ∂M , and since $w|_{\partial M} = 0$, it follows that $\nabla w = (\nabla w)^\perp = C\nu$ on ∂M . On the other hand,

$$1 = \Delta_f w = q_1 \langle \nabla w, \nu \rangle = \frac{A}{V} C \text{ implies } C = \frac{V}{A} \text{ and } |\nabla w| = \frac{V}{A}.$$

Therefore $\phi = \frac{1}{2} \left(\frac{V}{A} \right)^2$ on the boundary, and so we conclude by Proposition 2.3 that either

$$(3.10) \quad \phi = \frac{1}{2} \left(\frac{V}{A} \right)^2 \quad \text{in } M$$

or

$$(3.11) \quad \frac{\partial \phi}{\partial \nu}(y) > 0, \quad \forall y \in \partial M.$$

From $w|_{\partial M} = 0$, we have

$$\begin{aligned} 1 &= (\Delta_f w)|_{\partial M} = nH\eta + \text{Hess } w(\nu, \nu) - \frac{V}{A} \langle \nabla f, \nu \rangle \\ &= \frac{nV}{A} \left(H_f + \frac{\langle \nabla f, \nu \rangle}{n} \right) + \text{Hess } w(\nu, \nu) - \frac{V}{A} \langle \nabla f, \nu \rangle \\ &= \frac{nV}{A} H_f + \text{Hess } w(\nu, \nu). \end{aligned}$$

Hence it holds on ∂M that

$$\begin{aligned} \frac{\partial \phi}{\partial \nu} &= \frac{V}{A} \text{Hess } w(\nu, \nu) - \frac{V}{kA} \\ &= \frac{V}{A} \left(1 - \frac{nV}{A} H_f \right) - \frac{V}{kA} \\ &= n \frac{V}{A} \left(\frac{k-1}{kn} - H_f \frac{V}{A} \right), \end{aligned}$$

which shows that (3.11) is not true since $H_f(x_0) \geq \frac{(k-1)A}{knV}$. Therefore ϕ is constant on M . Since the Drift Laplacian of ϕ vanishes, we infer that equality must hold in (3.9) and that give us equality in the Proposition 2.1, and consequently $1 = \Delta_f w = \frac{k}{n+1} \Delta w$ and $\text{Hess } w = \frac{\Delta w}{n+1} \langle \cdot, \cdot \rangle$. The remainder of the proof follows a similar arguments as in proof of Theorem 1.2.

□

Proof of Theorem 1.4. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of the problem (1.5):

$$(3.12) \quad \begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Observe that w is not a constant. Otherwise, we would conclude from $w|_{\partial M} = 0$ that $w \equiv 0$. Set $\eta = \frac{\partial w}{\partial \nu}|_{\partial M}$; then $\eta \neq 0$. In fact, if $\eta = 0$ then

$$w|_{\partial M} = (\nabla w)|_{\partial M} = \frac{\partial^2 w}{\partial \nu^2} = 0$$

which implies that $(\Delta_f w)|_{\partial M} = 0$ and so $\Delta_f w = 0$ on M by the maximum principal, which in turn implies that $w = 0$. This is a contradiction.

Since $w|_{\partial M} = 0$, we have by the divergence theorem that

$$(3.13) \quad \int_M \langle \nabla w, \nabla (\Delta_f w) \rangle = - \int_M w \Delta_f^2 w = 0,$$

hence

$$(3.14) \quad \int_{\partial M} \Delta_f w \frac{\partial w}{\partial \nu} = \int_M \langle \nabla (\Delta_f w), \nabla w \rangle + \int_M (\Delta_f w)^2 = \int_M (\Delta_f w)^2.$$

Since $w|_{\partial M} = 0$, we have $\nabla w = \frac{\partial w}{\partial \nu} \nu$ and

$$(3.15) \quad \begin{aligned} (\Delta_f w)|_{\partial M} &= \frac{\partial^2 w}{\partial \nu^2} + n H_f \frac{\partial w}{\partial \nu} - \langle \nabla f, \nabla w \rangle \\ &= q_1 \frac{\partial w}{\partial \nu} + n H_f \frac{\partial w}{\partial \nu} + \langle \nabla f, \nu \rangle \frac{\partial w}{\partial \nu} - \langle \nabla f, \nu \rangle \frac{\partial w}{\partial \nu} \\ &= q_1 \frac{\partial w}{\partial \nu} + n H_f \frac{\partial w}{\partial \nu} \end{aligned}$$

using (3.14) and (3.15) we obtain that

$$q_1 = \frac{\int_M (\Delta_f w)^2 - n \int_{\partial M} H_f \eta^2}{\int_{\partial M} \eta^2}.$$

On the other hand, substituting w into (2.5), we obtain

$$(3.16) \quad \begin{aligned} \frac{k-1}{k} \int_M (\Delta_f w)^2 &= \int_M \text{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} n H_f \eta^2 \\ &\geq \int_{\partial M} n H_f \eta^2, \end{aligned}$$

that is,

$$\int_M (\Delta_f w)^2 - \int_{\partial M} n H_f \eta^2 \geq \frac{n}{k-1} \int_{\partial M} H_f \eta^2 \geq c \int_{\partial M} \eta^2.$$

By expression for q_1 and estimate above, we obtain the desired estimate

$$(3.17) \quad q_1 \geq c.$$

Assume now that $q_1 = c$. So all inequalities in (3.16) become equalities. Thus, by Proposition 2.1, we have that

$$(3.18) \quad \text{Hess } w = \frac{\Delta w}{n+1} \langle \cdot, \cdot \rangle \quad \text{and} \quad \Delta_f w = -\frac{k}{k-n-1} \langle \nabla f, \nabla w \rangle.$$

Choice an orthonormal frame $\{e_1, \dots, e_n\}$ on M so that restricted to ∂M , $e_n = \nu$. On the other side, to $i = 1, \dots, n-1$, using that $w|_{\partial M} = 0$, we obtain

$$\begin{aligned} 0 &= \text{Hess } w(e_i, e_n) = e_i e_n(w) - \nabla_{e_i} e_n(w) \\ &= e_i(\eta) - \langle \nabla_{e_i} e_n, e_n \rangle \eta = e_i(\eta), \end{aligned}$$

follow that $\eta = b_0 = \text{const}$. Since (3.17) takes equality and η is constant, we conclude that $H_f = \frac{k-1}{n} c$, which implies from (3.15) that $(\Delta_f w)|_{\partial M} = k c b_0$, therefore, by maximum principle $\Delta_f w$ is constant on M which implies from

(3.18) that Δw is constant on M . The remainder of the proof follows a similar arguments as in proof of Theorem 1.2. \square

4. ESCOBAR TYPE THEOREM FOR THE STEKLOFF PROBLEM

Recall the Bochner type formula for weighted Riemannian manifold, which says: Any smooth function u on M holds that

$$(4.1) \quad \frac{1}{2}\Delta_f|\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla(\Delta_f u) \rangle + \text{Ric}_f(\nabla u, \nabla u).$$

An immediate consequence of the Bochner type formula is the result below, however we believe that this is not a sharp estimate.

Theorem 4.1. *Let M^{n+1} , $n \geq 2$ be a compact weighted Riemannian manifold with boundary ∂M . Assume that $\text{Ric}_f \geq 0$, $H_f \geq 0$ and that the second fundamental form satisfies $II \geq cI$ on ∂M , $c > 0$. Then*

$$p_1 > \frac{c}{2}.$$

Proof. Set $h = \frac{\partial u}{\partial \nu}$, and $z = u|_{\partial M}$ where u is solution of problem (1.3). We have $p_1 z = p_1 u = h$, thus $p_1 \bar{\nabla} z = \bar{\nabla} h$. By (2.4), we have

$$\begin{aligned} 0 > - \int_M |\text{Hess } u|^2 &\geq \int_M [(\Delta_f u)^2 - |\text{Hess } u|^2 - \text{Ric}_f(\nabla u, \nabla u)] \\ &= \int_{\partial M} [n H_f h^2 + 2h \bar{\Delta}_f z + II(\bar{\nabla} z, \bar{\nabla} z)] \\ &\geq -2 \int_{\partial M} \langle \bar{\nabla} h, \bar{\nabla} z \rangle + c \int_{\partial M} |\bar{\nabla} z|^2 \\ &\geq -2p_1 \int_{\partial M} |\bar{\nabla} z|^2 + c \int_{\partial M} |\bar{\nabla} z|^2 \end{aligned}$$

Note that

$$\int_{\partial M} |\bar{\nabla} z|^2 > 0.$$

Otherwise z is constant on the Boundary and hence f is constant on M which is a contradiction. Thus $p_1 > \frac{c}{2}$. \square

Below we present the proof of a sharp estimate of the non-zero first Stekloff eigenvalue on surfaces. The technique was introduced by Escobar in [4], and just enable us to attack this problem in context of surfaces.

Proof of Theorem 1.5. Let ϕ be a non-constant eigenfunction for the Stekloff problem (1.3). Consider the function $v = \frac{1}{2}|\nabla \phi|^2$, then by (4.1)

$$\Delta_f v = |\text{Hess } \phi|^2 + \langle \nabla \phi, \nabla(\Delta_f \phi) \rangle + \text{Ric}_f(\nabla \phi, \nabla \phi).$$

Since ϕ is a f -harmonic function and $\text{Ric}_f \geq 0$ we find that

$$(4.2) \quad \Delta_f v = |\text{Hess } \phi|^2 + \text{Ric}_f(\nabla \phi, \nabla \phi) \geq 0.$$

Therefore the maximum of v is achieved at some point $P \in \partial M$. The Proposition 2.3 implies that $(\partial v / \partial \eta)(P) > 0$ or v is identically constant.

Let's assume $(\partial v / \partial \eta)(P) > 0$ and let (t, x) be Fermi coordinates around the point P , that is, x represents a point on the curve ∂M and t represents the distance to the boundary point x . The metric has the form

$$(4.3) \quad ds^2 = dt^2 + h^2(t, x)dx^2,$$

where $h(P) = 1$, $(\partial h / \partial x)(P) = 0$. Thus

$$|\nabla \phi|^2 = \left(\frac{\partial \phi}{\partial t} \right)^2 + h^{-2} \left(\frac{\partial \phi}{\partial x} \right)^2,$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x \partial t} + h^{-2} \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} - h^{-3} \frac{\partial h}{\partial x} \left(\frac{\partial \phi}{\partial x} \right)^2.$$

Evaluating at the point P we obtain

$$(4.4) \quad \frac{\partial v}{\partial x}(P) = \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} = 0.$$

The f -Laplacian with respect to the metric given by (4.3) in Fermi coordinates (t, x) is

$$\Delta_f = \frac{\partial^2}{\partial t^2} + h^{-1} \frac{\partial h}{\partial t} \frac{\partial}{\partial t} + h^{-1} \frac{\partial}{\partial x} \left(h^{-1} \frac{\partial}{\partial x} \right) - \frac{\partial f}{\partial t} \frac{\partial}{\partial t} - h^{-2} \frac{\partial f}{\partial x} \frac{\partial}{\partial x}.$$

The geodesic curvature of ∂M can be calculated in terms of the function f and its first derivative as follows:

$$(4.5) \quad \begin{aligned} k_g &= - \left\langle \nabla_{\partial/\partial x} \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right\rangle = - \left\langle \nabla_{\partial/\partial t} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle \\ &= - \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = - \frac{1}{2} \frac{\partial}{\partial t} (h^2) = -hh'. \end{aligned}$$

Hence at P we find that

$$(4.6) \quad 0 = \Delta_f \phi = \frac{\partial^2 \phi}{\partial t^2} - k_g \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial f}{\partial t} \frac{\partial \phi}{\partial t} - \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial x}.$$

Using the equality (4.5) we get that

$$(4.7) \quad \frac{\partial v}{\partial t}(P) = \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial t \partial x} + k_g \left(\frac{\partial \phi}{\partial x} \right)^2.$$

Multiplying the equation (4.6) by $-\frac{\partial \phi}{\partial t}$ and adding with the equation (4.7) we obtain

$$(4.8) \quad \frac{\partial v}{\partial t}(P) = k_g |\nabla \phi|^2 - \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial t \partial x} + \frac{\partial f}{\partial t} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t}.$$

If $\frac{\partial\phi}{\partial x}(P) \neq 0$, the equation (4.4) and the boundary condition yields

$$(4.9) \quad \frac{\partial^2\phi}{\partial x^2}(P) = p_1 \frac{\partial\phi}{\partial t}(P).$$

Therefore the equation (4.8) can be re-written using the boundary condition as

$$(4.10) \quad \frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla\phi|^2 + p_1 \left(\frac{\partial\phi}{\partial x} \right)^2 + \frac{\partial\phi}{\partial x} \frac{\partial^2\phi}{\partial t\partial x} + \frac{\partial f}{\partial t} \left(\frac{\partial\phi}{\partial t} \right)^2 + \frac{\partial f}{\partial x} \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial t}.$$

Notice that by (4.4) we obtain, using (4.9),

$$(4.11) \quad 0 = \frac{\partial\phi}{\partial t} \frac{\partial^2\phi}{\partial x\partial t} + \frac{\partial\phi}{\partial x} \frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial t} \left(\frac{\partial^2\phi}{\partial x\partial t} + p_1 \frac{\partial\phi}{\partial x} \right),$$

that is,

$$p_1 \frac{\partial\phi}{\partial x} = -\frac{\partial^2\phi}{\partial x\partial t}.$$

Thus (4.10) becomes

$$\frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla\phi|^2 + \frac{\partial f}{\partial t} \left(\frac{\partial\phi}{\partial t} \right)^2 + \frac{\partial f}{\partial x} \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial t}$$

and we write

$$\frac{\partial v}{\partial t}(P) = (k_g - p_1)|\nabla\phi|^2 + \frac{\partial\phi}{\partial t} \langle \nabla\phi, \nabla f \rangle.$$

Since $f|_{\partial M}$ is constant, so $\frac{\partial f}{\partial x}(P) = 0$, and using that ν coincide with normalized gradient of f in ∂M we have that $\frac{\partial f}{\partial t} \leq 0$

$$\begin{aligned} \frac{\partial v}{\partial t}(P) &= (k_g - p_1)|\nabla\phi|^2 + \frac{\partial f}{\partial t} \left(\frac{\partial\phi}{\partial t} \right)^2 \\ &\geq \left(k_g + \frac{\partial f}{\partial t} - p_1 \right) |\nabla\phi|^2, \end{aligned}$$

hence

$$(4.12) \quad (k_g + \frac{\partial f}{\partial t} - p_1)|\nabla\phi|^2 < 0,$$

and $p_1 > k_g + \frac{\partial f}{\partial t} = k_g - f_\nu \geq c$.

Now we assume that $\frac{\partial\phi}{\partial x}(P) = 0$. A straighforward calculation yields

$$\frac{\partial^2 v}{\partial x^2}(P) = \left(\frac{\partial^2\phi}{\partial x\partial t} \right)^2 + \frac{\partial\phi}{\partial t} \frac{\partial^3\phi}{\partial x^2\partial t} + \left(\frac{\partial^2\phi}{\partial x^2} \right)^2.$$

Using the boundary condition we get that

$$(4.13) \quad \frac{\partial^2 v}{\partial x^2}(P) = p_1^2 \phi \frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial^2\phi}{\partial x^2} \right)^2 \leq 0.$$

Since $\frac{\partial \phi}{\partial x}(P) = 0$, the equation (4.8) implies that

$$\frac{\partial v}{\partial t}(P) = k_g \left(\frac{\partial \phi}{\partial t} \right)^2 + p_1 \phi \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial f}{\partial t} \left(\frac{\partial \phi}{\partial t} \right)^2 = \left(k_g + \frac{\partial f}{\partial t} \right) p_1^2 \phi^2 + p_1 \phi \frac{\partial^2 \phi}{\partial x^2}.$$

Thus

$$(4.14) \quad \left(k_g + \frac{\partial f}{\partial t} \right) p_1^3 \phi^2 + p_1^2 \phi \frac{\partial^2 \phi}{\partial x^2} < 0.$$

Adding inequality (4.13) with (4.14) we obtain

$$\left(\frac{\partial^2 \phi}{\partial x^2} + p_1^2 \phi \right)^2 + p_1^3 \left(k_g + \frac{\partial f}{\partial t} - p_1 \right) \phi^2 < 0.$$

Hence

$$p_1 > k_g + \frac{\partial f}{\partial t} = k_g - f_\nu \geq c.$$

Let's assume that v is the constant function. Observe that $v \not\equiv 0$ because ϕ is non-constant. Since v is f -harmonic, inequality (4.2) implies that

$$\text{Hess } \phi = 0 \quad \text{and} \quad \text{Ric}_f(\nabla \phi, \nabla \phi) = 0, \quad \text{on } M.$$

Now, using that $\Delta_f \phi = 0$, we obtain that $\langle \nabla \phi, \nabla f \rangle = 0$ and hereby $\text{Hess } f(\nabla \phi, \nabla \phi) = 0$. Thus, the Gaussian curvature K of M vanishes. Moreover, using the structure of surfaces,

$$(4.15) \quad \nabla f = \lambda J(\nabla \phi),$$

where J is the anti-clockwise rotation of $\pi/2$ in the tangent plane.

Let $\{e_1, e_2\}$ be a local orthonormal frame field such that e_1 is tangent to ∂M and $e_2 = \eta$. So,

$$\begin{aligned} 0 &= \text{Hess } \phi(e_1, e_2) = e_1 e_2(\phi) - \nabla_{e_1} e_2(\phi) \\ &= e_1(p_1 \phi) - \langle \nabla_{e_1} e_2, e_1 \rangle \phi_1 \\ &= (p_1 - k_g) \phi_1. \end{aligned}$$

Observe that if $\phi_1 = 0$ on ∂M , then $\phi = \text{constant}$ on ∂M and hence ϕ is a constant function on M which is a contradiction. Thus $p_1 = k_g$ except maybe when $\phi_1 = 0$. Since $\text{Hess } \phi(e_1, e_1) = 0$ we have

$$\begin{aligned} 0 &= \text{Hess } \phi(e_1, e_1) = e_1 e_1(\phi) - \nabla_{e_1} e_1(\phi) \\ &= e_1(e_1 \phi) - \langle \nabla_{e_1} e_1, e_2 \rangle e_2(\phi) \\ &= e_1(e_1 \phi) + k_g p_1 \phi. \end{aligned}$$

Hence ϕ satisfies on the boundary a second order differential equation

$$(4.16) \quad \begin{aligned} \frac{d^2 \phi}{dx^2} + k_g p_1 \phi &= 0 \\ \phi(0) &= \phi(\ell) \end{aligned}$$

where ℓ represents the length of ∂M . The function ϕ does not vanishes identically, thus $\phi_1 = 0$ except for a finite number of points. Therefore

$p_1 = k_g$ except for a finite number of points and using the continuity of k_g , we conclude that $p_1 = k_g$ everywhere. Therefore,

$$p_1 = k_g - f_\nu + f_\nu \geq c,$$

and the equality between p_1 and c occurs if $k_g = k_0$ and $f_\nu = 0$. Using $K = 0$ and k_g is a positive constant, we conclude that M is an Euclidean ball.

Furthermore, the identity (4.15) and after a straightforward computations we obtain that

$$\text{Hess } f = \frac{\Delta f}{2v} (J(\nabla \phi) \otimes J(\nabla \phi)).$$

It easy to see, using that M is an Euclidean ball, that $\phi = x_i$, that is, ϕ is a coordinate function. Thus, using the expression of ϕ , f satisfies $\text{Hess } f = 0$ and as f is constant on the boundary, we have f constant.

□

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